Since the neutrinos only interact with the down quarks in the nucleons through the exchange of down-type squarks, constraint (18) translates into

$$(1+2Y_n)G^d_{e\alpha} \gtrsim 10^{-2}\frac{G_F}{2}, \quad (27)$$

so that the couplings involved should satisfy $|\lambda'\lambda'| \gtrsim 10^{-3}(m_{\tilde q}/100\ \text{GeV})^2$. In the case of $\nu_e\nu_\mu$ conversion, scattering through $\tilde s_L$ exchange involves the product of couplings $\lambda'_{121}\lambda'_{221}$, while the scattering through $\tilde b_L$ exchange involves $\lambda'_{131}\lambda'_{231}$. However, since these couplings also induce the process $\mu \to e\gamma$, they are very suppressed and $\nu_e\nu_\mu$ conversion is not allowed. Instead, it is possible to generate $\nu_e\nu_\tau$ conversion by exchange of either left or right down-type squarks, since there are no strong bounds on λ_{3jk} alone, while the bound from $\tau \to e\gamma$ is [15] $\lambda'_{1jk}\lambda'_{3jk} \lesssim 5\times10^{-2}(m_{\tilde q}/100\ \text{GeV})^2$. It is interesting to note that for this model the required couplings could be probed at a τ factory [15].

In conclusion, in the same way as small neutrino mixing in a vacuum can be amplified producing significant oscillations of the neutrinos that cross a resonance layer while propagating in a medium, we have shown that similar effects can be obtained in the presence of flavor-changing neutrino interactions. This has important applications to solar neutrinos, since allowed strengths for those new interactions can lead to a solution to the solar-neutrino deficit even for negligibly small vacuum mixings.

We have shown how lepton-number-violating couplings that can be present in the minimal supersymmetric extension of the standard model are able to generate the required flavor-nondiagonal neutrino interactions with quarks and leptons, as well as the necessary neutrino masses, taking into account the experimental bounds on the new couplings.

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[1] L. Wolfenstein, Phys. Rev. D **17**, 2369 (1978).
[2] S. P. Mikheyev and A. Yu. Smirnov, Yad. Fiz. **42**, 1441 (1985) [Sov. J. Nucl. Phys. **42**, 913 (1985)].
[3] H. A. Bethe, Phys. Rev. Lett. **56**, 1305 (1986); S. P. Rosen and J. M. Gelb, Phys. Rev. D **34**, 969 (1986).
[4] S. J. Parke, Phys. Rev. Lett. **57**, 1275 (1986).
[5] For reviews, see S. M. Bilenky and S. T. Petcov, Rev. Mod. Phys. **59**, 671 (1987); T. K. Kuo and J. Pantaleone, *ibid.* **61**, 937 (1989).
[6] R. Davis, in *Neutrino '88*, Proceedings of the 13th International Conference on Neutrino Physics and Astrophysics, Boston, Massachusetts, 1988, edited by J. Schneps, T. Kafka, W. A. Mann, and P. Nath (World Scientific, Singapore, 1989), p. 518.
[7] K. S. Hirata *et al.*, Phys. Rev. Lett. **65**, 1297 (1990); **65**, 1301 (1990).
[8] Effects of a nonuniversal diagonal strength of the neutral currents have been discussed by J. Valle, Phys. Lett. B **199**, 432 (1987).
[9] For reviews, see, e.g., H. P. Nilles, Phys. Rep. **110**, 1 (1984); H. E. Haber and G. L. Kane, *ibid.* **117**, 75 (1985).
[10] C. Aulak and R. Mohapatra, Phys. Lett. **119B**, 136 (1983); F. Zwirner, *ibid.* **132B**, 103 (1983); L. J. Hall and M. Suzuki, Nucl. Phys. **B231**, 419 (1984); I. H. Lee, *ibid.* **B248**, 120 (1984); J. Ellis *et al.*, Phys. Lett. **150B**, 142 (1985); S. Dawson, Nucl. Phys. **B261**, 297 (1985); R. Barbieri and A. Masiero, *ibid.* **B267**, 679 (1986); S. Dimopoulos and L. J. Hall, Phys. Lett. B **207**, 210 (1987).
[11] R. Barbieri *et al.*, Phys. Lett. B **252**, 251 (1990).
[12] E. Roulet and D. Tommasini, Phys. Lett. B **256**, 218 (1991).
[13] V. Barger, G. F. Giudice, and T. Y. Han, Phys. Rev. D **40**, 2987 (1989).
[14] Some caution should be taken with the bounds quoted because $\lambda_{121} < 0.04$ is a 1σ bound, while $\lambda_{231} < 0.09$ is a 2σ bound. At 1σ there is no allowed value for λ_{231}, while at 2σ the bound on λ_{121} is weaker.
[15] A. Masiero, Report No. DFPD/90/TH732 (unpublished).

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Covariant description of the canonical formalism

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In a gauge theory, one can define the Poisson brackets of gauge-invariant functions ("observables") by three different methods. The first method is based on the constrained Hamiltonian reformulation of the theory. The other two methods, namely, the Peierls method and the covariant symplectic approach, deal directly with the Lagrangian. It is explicitly shown that these three methods are equivalent for an arbitrary gauge theory. The equivalence proof relies on the invariance of the Poisson structure among the observables under the introduction of auxiliary fields.

The physical quantities of a gauge theory ("observables") are functions defined on the reduced phase space. This space can be described as the quotient of the space of solutions of the equations of motion by the gauge transformations (Refs. [1-3]). Thus, if $S[\phi^i]$ and $\delta\phi^i(x) = \int d^nx' R^i_\alpha(x,x')\varepsilon^\alpha(x')$ are, respectively, the action and the gauge transformations, an observable can be thought of as a functional $A[\phi^i]$ of the histories that is on-shell gauge invariant, i.e.,

$$\int d^nx\, R^i_\alpha(x,x')\delta A/\delta\phi^i(x) = \int d^nx\, \lambda^i_\alpha(x,x')\delta S/\delta\phi^i(x), \quad (1)$$

for some $\lambda^i_\alpha(x,x')$, with the understanding that two solutions of (1) that coincide on shell should be identified. The reduced phase space of a relativistic gauge theory is a relativistic concept since the equations of motion and the gauge transformations are then relativistically invariant. For that reason, one also uses the terminology "covariant phase space." The purpose of this Rapid Communication is to establish the equivalence of the various methods for defining a Poisson-brackets structure among the observables: namely, the Hamiltonian approach, the Peierls method, and the covariant symplectic approach.

The standard Hamiltonian method is based on a definite choice of the spacetime observer and proceeds as follows. If the Lagrangian contains the fields and their time derivatives up to first order (Ref. [4]), the initial data for the field equations can be taken to be the fields and their first-order time derivatives on the hypersurface $x^0 = 0$. These initial data are, however, neither independent nor physically distinct. This is because the equations of motion imply some constraints on the ϕ's and $\dot\phi$'s. Furthermore, different admissible sets of allowed ϕ's and $\dot\phi$'s may lead to solutions of the equations of motion that are related by a gauge transformation. In order to get coordinates on the reduced phase space, one needs to solve the constraint equations and to factor out the action of the gauge transformations on the initial data.

To that end, one introduces auxiliary fields, which are (i) the momenta $\pi_i = \delta L/\delta\dot\phi^i$ conjugate to the ϕ's (these are subject to the primary constraints $G_{m_1}[\phi^i,\pi_i] = 0$ (if any)) and (ii) the Lagrange multipliers u^{m_1} associated with the primary constraints. With these variables, the action can be rewritten as

$$S[\phi^i,\pi_i,u^{m_1}] = \int dx^0 \int d^{n-1}x(\pi_i\dot\phi^i - \mathcal{H} - u^{m_1}G_{m_1}). \quad (2)$$

The equations of motion $\delta S/\delta\pi_i(x) = 0$, $\delta S/\delta u^{m_1}(x) = 0$ can be solved for π_i and u^{m_1}. Upon elimination of π_i and u^{m_1} by means of their equations of motion, one gets the original action $S[\phi^i]$ back, hence, the terminology "auxiliary fields" for π_i and u^{m_1} (Refs. [5] and [6]).

The constraints $G_{m_1} \approx 0$ are, in general, not the only ones in the theory. The consistency conditions $\dot G_{m_1} \approx 0$ imply further constraints. The complete set of constraints can be separated into "first-class constraints" $\gamma_a \approx 0$ and "second-class constraints" $\chi_\alpha \approx 0$ (Ref. [7]). The second-class constraints are such that the Poisson-brackets matrix $[\chi_\alpha,\chi_\beta]$ can be inverted, while the first-class constraints satisfy $[\gamma_a,\gamma_b] \approx 0$ and $[\gamma_a,\chi_\alpha] \approx 0$.

One can choose the Hamiltonian to be first class. One then finds that the Lagrange multipliers associated with the second-class constraints are restricted to be zero, while those associated with the first-class constraints are not determined by the equations of motion and can be given arbitrary values by a gauge transformation. Hence, the physically distinct initial data are to be found among the ϕ^i's and the π_i's at a given time subject to the constraint equations $\gamma_a \approx 0$ and $\chi_\alpha \approx 0$. The action of the gauge transformations on these initial data can be shown under quite general conditions to be generated by all the first-class constraints $\gamma_a \approx 0$ (Refs. [5], [6], and [8]). Consequently, the reduced phase space is isomorphic to the quotient $\Sigma/\mathcal{G}$ of the constraint surface $\Sigma: \gamma_a \approx 0, \chi_\alpha \approx 0$ by the gauge orbits $\mathcal{G}$ generated by the first-class constraints. On this quotient space, there is a natural brackets structure, which is the inverse of the invertible closed two-form induced on $\Sigma/\mathcal{G}$ by the phase-space canonical symplectic structure $\int d^{n-1}x\, \delta\pi_i \wedge \delta\phi^i$ (see Ref. [6], Chap. 2).

This brackets structure is known as the Dirac brackets because it coincides with the brackets defined by Dirac (Ref. [5]) when the observables are rewritten as functions of the canonical variables at a given time by using the equations of motion (including the constraints) (Ref. [6], Chap. 2). Given two observables A and B, one can add to them appropriate combinations $\lambda^\alpha\chi_\alpha$ of the second-class constraints such that $[A,\chi_\alpha] \approx 0$ and $[B,\chi_\alpha] \approx 0$ (in addi-

[illegible] $\pi_a\} = 0$). With this choice, the [illegible] coincide with the Poisson brackets [illegible] the π's:

$$[illegible] \left[\frac{\delta A}{\delta\phi^i(\mathbf{x})} \frac{\delta B}{\delta\pi_i(\mathbf{x})} - \frac{\delta A}{\delta\pi_i(\mathbf{x})} \frac{\delta B}{\delta\phi^i(\mathbf{x})} \right] . \tag{3}$$

Two alternative proposals have been made for deriving the brackets structure among the observables without having to go through the nonmanifestly covariant Hamiltonian steps. The first one follows a method due to Peierls (Refs. [9] and [10]). The second one computes the reduced phase-space symplectic structure directly from the Lagrangian (Refs. [1–3]).

In the Peierls method, the brackets $[A,B]$ of two observables (localized in time) are obtained by observing that a modification of the action by means of A, $S \to S + \varepsilon A$, produces a modification of the solutions of the equations of motion. Let $\phi^i(x)$ be a given solution of the unperturbed problem and $\phi^i(x) + \varepsilon\delta_A^- \phi^i(x)$ be a solution of the modified variational principle that reduces to $\phi^i(x)$ up to a gauge transformation in the remote past. The perturbation $\delta_A^- \phi^i(x)$ of the solution $\phi^i(x)$ is defined up to a gauge transformation. If one replaces $\phi^i(x)$ by $\phi^i(x) + \varepsilon\delta_A^- \phi^i(x)$, the on-shell value of the observable B gets modified as $B \to B + \varepsilon D_A B$, with

$$D_A B = \int [\delta B/\delta\phi^i(x)] \delta_A^- \phi^i(x) d^n x . \tag{4}$$

The ambiguity in $\delta_A^- \phi^i(x)$ does not affect B, which is on-shell gauge invariant. The Peierls brackets are given by

$$[A,B] = D_A B - D_B A . \tag{5}$$

The Peierls brackets are defined for ϕ^i on-shell and for gauge-invariant functions [otherwise, $D_A B$ in (4) is ambiguous]. Furthermore, if A is replaced by $A' = A + \int \lambda^i(x) [\delta S/\delta\phi^i(x)] d^n x$, with λ^i localized in time, then $D_{A'} B = D_A B - \int \lambda^i(x) [\delta B/\delta\phi^i(x)] d^n x$ and $D_B A' = D_B A - \int \lambda^i(x) [\delta B/\delta\phi^i(x)] d^n x$. That is,

$$\left[\int \lambda^i(x) [\delta S/\delta\phi^i(x)] d^n x , B \right] = 0 . \tag{6a}$$

Similarly,

$$\left[A, \int \lambda^i(x) [\delta S/\delta\phi^i(x)] d^n x \right] = 0 . \tag{6b}$$

Hence, the Peierls brackets are defined in the reduced phase space.

The symplectic approach to the reduced phase space developed in Refs. [1–3] proceeds differently. The variation of the action

$$S = \int \mathcal{L}(\phi^i, \partial_\mu\phi^i, \ldots, \partial_{\mu_1} \cdots \partial_{\mu_k}\phi^i) d^n x \tag{7}$$

is equal to

$$\delta S = \int \left[\frac{\delta \mathcal{L}}{\delta\phi^i} \delta\phi^i + \partial_\mu j^\mu \right] d^n x , \tag{8}$$

where

$$j^\mu = \frac{\delta \mathcal{L}}{\delta(\partial_\mu\phi^i)} \delta\phi^i + \cdots + \frac{\delta \mathcal{L}}{\delta(\partial_{\mu\mu_2 \cdots \mu_k}\phi^i)} \partial_{\mu_2 \cdots \mu_k} \delta\phi^i . \tag{9}$$

Here, the variational derivatives $\delta/\delta a$ of a local function are defined by

$$\frac{\delta}{\delta a} = \frac{\partial}{\partial a} - \partial_\mu \frac{\partial}{\partial(\partial_\mu a)} + \partial_\mu\partial_\nu \frac{\partial}{\partial(\partial_\mu\partial_\nu a)} - \cdots . \tag{10}$$

The variation δS can be viewed as an exact one-form in the space of all histories. Using $\delta^2 = 0$, one can transform (8) into the identity

$$\int \left[\delta \left[\frac{\delta \mathcal{L}}{\delta\phi^i} \delta\phi^i \right] + \partial_\mu \delta j^\mu \right] d^n x = 0 . \tag{11}$$

On the stationary surface (space of solutions of the equations of motion), the pullback of (11) reduces to the boundary term

$$\int \partial_\mu (\delta j^\mu) d^n x = 0 . \tag{12}$$

Hence, the flux of the symplectic current

$$\int_\sigma \delta j^\mu d\Sigma_\mu \tag{13}$$

extended over a spacelike hypersurface σ defines a two-form on the stationary surface that does not depend on the choice of σ. (We assume the fields decrease fast enough at spacelike infinity.) That two-form is closed and annihilated by the gauge transformations (Ref. [11]). One can thus take the quotient of the stationary surface by the gauge transformations and obtain from (13) the symplectic two-form of the reduced phase space introduced in Refs. [1–3].

To show the equivalence of definitions (5) and (13) with the Hamiltonian analysis, it is necessary to prove the invariance of (5) and (13) under the introduction of auxiliary fields. Consider, then, an action $S[y^i, z^a]$ depending on fields y^i and "auxiliary fields" z^a, i.e., assume that the equations $\delta S/\delta z^a(x) = 0$ can be solved to yield z^a as a function of y^i and its derivatives:

$$\delta S/\delta z^a(x) = 0 \to z^a = Z^a(y^i, \partial_\mu y^i, \ldots, \partial_{\mu_1 \cdots \mu_k} y^i) . \tag{14}$$

Let $\bar{S}[y^i]$ be the action obtained by eliminating the auxiliary fields:

$$\bar{S}[y^i] = S[y^i, Z^a] . \tag{15}$$

The equations of motion $\delta\bar{S}/\delta y^i(x) = 0$ and $[\delta S/\delta y^i(x)](y^i, Z^a) = 0$ are equivalent so that the space of solutions of the equations of motion for the theories based on $\bar{S}[y^i]$ and $S[y^i, z^a]$ are identical. The concepts of observables are also equivalent because the gauge transformations for y^i can be taken to be the same in both theories (Ref. [12]) and because there is at least one observable that involves only y^i in any equivalence class of observables $A[y^i, z^a]$ for $S[y^i, z^a]$:

$$\bar{A}[y^i] \equiv A[y^i, Z^a]$$

$$= A[y^i, z^a] + \int [illegible] d^n x . \tag{16}$$

(Since $A[y^i, Z^a]$ and [illegible] on (14), they differ by equations-of-[illegible].)

Given two observables [illegible] $B[y^i, z^a]$, one can compute their brackets [illegible] (5) with the help of the action $S[y^i, z^a]$, [illegible] the result, which is defined for $y^i(x)$ and $z^a(x)$ on shell, by $[A,B]_S$. Similarly, one can compute the brackets of $\bar{A}[y^i, Z^a]$ and $\bar{B}[y^i, Z^a]$ with the help of $\bar{S}$. We denote the result by $[\bar{A}, \bar{B}]_{\bar{S}}$.

Theorem 1.

$$[A,B]_S = [\bar{A}, \bar{B}]_{\bar{S}} . \tag{17}$$

Proof. The proof is immediate. Because of (16) and (6), one can assume A and B involve $y^i(x)$ only, in which case $A = \bar{A}$ and $B = \bar{B}$. The equations of motion following from $S + \varepsilon A$ imply then $z^a = Z^a$ with the same Z^a, since $\delta A/\delta z^a = 0$. Hence, the equations of motion for y^i derived from $S + \varepsilon A$ and $\bar{S} + \varepsilon A$ are equivalent, so that the perturbations $D_A B$ and $\bar{D}_A B$ computed with S or with $\bar{S}$ are equal. This yields (17).

Turn now to the symplectic two-form (13). Again, one can define two symplectic currents: One δj^μ for the action S; and one $\delta\bar{j}^\mu$ for the action $\bar{S}$. The two-forms $\int_\sigma \delta j^\mu d\Sigma_\mu$ and $\int_\sigma \delta\bar{j}^\mu d\Sigma_\mu$ are defined in the same space on shell and one must prove that they are equal. This is the content of the following theorem.

Theorem 2.

$$\int_\sigma \delta j^\mu d\Sigma_\mu = \int_\sigma \delta\bar{j}^\mu d\Sigma_\mu . \tag{18}$$

Proof. The proof is again immediate. Because $S = \bar{S}$ when $\delta S/\delta z^a(x) = 0$, the surface terms in the variations of S and $\bar{S}$ are equal under the same conditions, i.e.,

$$\int_\sigma j^\mu d\Sigma_\mu = \int_\sigma \bar{j}^\mu d\Sigma_\mu , \tag{19}$$

where in j^μ, δz^a is replaced by δZ^a. The equality (19) implies (18).

To complete the proof of the equivalence of the various brackets structures that can be defined in the reduced phase space, it remains to show that the Peierls brackets (5) and the symplectic structure (13) coincide with the corresponding Hamiltonian concepts in the particular case of the first-order action (2). This is obvious for the symplectic structure (13) because the flux of the symplectic current through $x^0 =$ const gives, in that case,

$$\int d^{n-1}x \, \delta\pi_i \wedge \delta\phi^i , \tag{20}$$

which is the standard phase-space symplectic structure. There is no contribution from the Lagrange multipliers since these are undifferentiated in (2). Structure (20) induces, as we have recalled, the Hamiltonian brackets in the reduced phase space.

Similarly, the addition to the action (2) of the first-class functional $A[\phi^i(x), \pi_i(x)]$ depending on the fields and their momenta at a given time, say $t = 0$, amounts to replacing the first-class Hamiltonian $\int d^{n-1}x \mathcal{H}$ with the first-class Hamiltonian $\int d^{n-1}x \mathcal{H} - \varepsilon A \delta(t)$. This implies that $D_A B$ is equal to the Poisson brackets $[A,B]$ if B is a first-class function depending on the canonical variables at time $t' > 0$, and $D_A B = 0$ if $t' < 0$ (Ref. [13]). Even though the perturbations $D_A u^{m_1}$ of the multipliers are ill defined, the perturbation $D_A B$ is unambiguous because B is first class. Similarly $D_B A = -[A,B]$ if $t' < 0$ and $D_B A = 0$ if $t' > 0$. Hence, the Peierls definition gives the ordinary Poisson brackets (3) among first-class functions.

We have thus established that the various definitions of the Poisson brackets for the observables of an arbitrary gauge theory are equivalent. Crucial in the proof is the property that the passage to the canonical formalism amounts to introducing auxiliary fields. It is thus quite useful to realize that the conjugate momenta and the Lagrange multipliers u^{m_1} can be eliminated from the action (2) by means of their own equations of motion.

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[1] E. Witten, Nucl. Phys. **B276**, 291 (1986); C. Crnković and E. Witten, in *300 Years of Gravitation*, edited by S. Hawking and W. Israel (Cambridge Univ. Press, Cambridge, England, 1987).

[2] G. Zuckerman, Yale University report (unpublished).

[3] A. Ashtekar, L. Bombelli, and O. Reula, in *Analysis, Geometry and Mechanics: 200 Years After Lagrange*, edited by M. Francaviglia and D. Holm (North-Holland, Amsterdam, 1990).

[4] By introducing auxiliary fields, one can assume without loss of generality that the Lagrangian depends on the coordinates and velocities only. For instance, the Lagrangian $L(q,\dot{q},\ddot{q})$ can be replaced by $L'(q,\dot{q},u,\dot{u},\lambda) = L(q,\dot{q},\dot{u}) - \lambda(u - \dot{q})$. The equations $\delta L'/\delta\lambda = 0$ and $\delta L'/\delta u = 0$ can be solved for u and λ. After elimination of λ and u from L', one gets $L(q,\dot{q},\ddot{q})$ back. As shown in the text, the introduction of auxiliary fields does not modify the Peierls brackets or the covariant symplectic structure introduced in Refs. [1–3].

[5] The Hamiltonian formulation for gauge theories is due to Dirac, see, e.g., P. A. M. Dirac, *Lectures on Quantum Mechanics* (Yeshiva University, New York, 1967); for the structure of the constrained surface, see also P. G. Bergmann and J. Goldberg, Phys. Rev. **98**, 531 (1955); A. Lichnerowicz, C. R. Acad. Sci. **280**, 523 (1975); M. J. Gotay, J. M. Nester, and G. Hinds, J. Math. Phys. **19**, 2388 (1978). We use in the text many results explained in Ref. [6].

[6] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems* (Princeton Univ. Press, Princeton, NJ, in press).

[7] Note that the constraints can all be written as combinations of the equations of motion following from (2).

[8] M. Henneaux, C. Teitelboim, and J. Zanelli, Nucl. Phys. **B332**, 169 (1990).

[9] R. E Peierls, Proc. R. Soc. London **A214**, 143 (1952).

[10] B. S. DeWitt, in *Relativity, Groups and Topology*, edited by C. DeWitt and B. S. DeWitt (Gordon and Breach, New York, 1964).

[11] The proof of this statement is not immediate within the covariant formalism and has been checked in particular instances in Refs. [1–3]. It is, however, a direct consequence of the equivalence theorem proved below and of the Hamiltonian analysis.

[12] M. Henneaux, Phys. Lett. B **238**, 299 (1990).

[13] In computing the Poisson brackets $[A,B]$ one must express the canonical variables at t' in terms of the canonical variables at $t = 0$. See Ref. [9].